JGP - Vol. 4, n. 2, 1987

A stability criterion for Hamiltonian systems with symmetry

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Abstract. *We find a simple criterion for orbital stability for the general hamiltonian systems with symmetry in the equivariant symplectic and in the corresponding Poisson context.*

INTRODUCTION

This paper presents a theorem which gives a simple criterion for the orbital stabilty and instabilty of the relative equilibria of the Hamiltonian system with symmetry and its version in the Poisson context.

It is well-known that for the generic Hamiltonian system, the only way of proving the stability of the equilibria is proving the strict convexity of the Hamiltonian at the equilibria. When the Hamiltonian has a symmetry and the equilibrium is not a fixed point of the symmetry group, then the Hessian always has some kernel. In this case, it is reasonable to investigate the stability question up to symmetry, i.e. orbital stability. This is equilivalent to investigating the stability of the equilibria in the reduced space.

However, in many practical problems, it is easier to work in the original space using the so called «Energy-Casimir method» [4], since it is not simple to parametrize the reduced space. When we apply the Energy-Casimir method, however, we have to deliberately choose the Casimirs for the stability and we have to carry

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Key-Words: Hamiltonian system, (relative) equilibria, orbital Stability, Lyapunov Stability, Poisson Structure, symplectic leaves, Casimir function, Energy-Casimir method.

1985 AMS Subject Classification: Primary 34D20, 58F10, 58F05, Secondary 70E15, 70H05.

out some extra work like the linearized analyses or finding some Lyapunov function for the instability.

In this paper, we will present a direct criterion without involving the analysis of the linearized Hamiltonian vector field for stability and instability of the relative equilibria, both in the S^1-equivariant symplectic context and Poisson context with codimension one regular symplectic leaves and generalize this to larger groups. This is motivated by works on non-linear wave and Schrödinger equations (See [2], [3], [5], [7] and [10], especially [3]). We will also apply this criterion to the free rigid body recovering the well-known stability criterion from classical mecahanics and to the planar coupled rigid bodies.

We may consider the result in this paper as a complementary work for Energy-Casimir method [4] and as a geometric adaptation of the one in [3].

We would like to thank Jerry Marsden and Alan Weinstein for their interest and helpful suggestions on this work and to W. Strauss for sending me a preprint of their paper [3].

1. STATEMENT OF THE RESULTS

Let (M, Ω, G, J) be a Hamiltonian G-space with G-invariant metric and H a G-invariant Hamiltonian function. (Here, Ω is the symplectic form on M and J is the moment mapping associated to the symplectic G-action). Then the Hamiltonian system,

$$\frac{dx}{dt} = X_H(x). \tag{1.1}$$

drops to the reduced space $J^{-1}(\mu)/G_\mu$, where G_μ is the isotropy group of μ. The equilibria of the reduced system are called *relative equilibria* of the system (1.1).

Since it is quite troublesome to parametrize the quotient space $J^{-1}(\mu)/G_\mu$, another description of the relative equilibria is often useful, especially for stability questions. Since J is invariant under the H-flow, the relative equilibria are given by the equation,

$$dH(x) = < \xi, dJ(x) > \tag{1.2}$$

for some $\xi \in g_\mu$ where g_μ is the Lie algebra of G_μ (See [1]).

Let x_ξ be a corresponding relative equilibrium. Then we have the following proposition.

PROPOSITION 1. *The curve $a(t) = \exp t\xi \cdot x_\xi$ gives a solution of the equation (1.1).*

Proof.

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\exp t\xi \cdot x_\xi \\ &= \exp t\xi \cdot \xi_G(x_\xi) \\ &= \exp t\xi \cdot X_{<\xi,J>}(x_\xi)\end{aligned}$$

by the definition of the moment map. However, we get $X_{<\xi,J>}(x_\xi) = X_H(x_\xi)$ from equation (1.2). Therefore,

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t}(t) &= \exp t\xi \cdot X_H(x_\xi) \\ &= \exp t\xi \cdot X_H(\exp - t\xi \cdot \exp t\xi \cdot x_\xi) \\ &= X_H(\exp t\xi \cdot x_\xi) \quad \text{(from the equivariance of } H) \\ &= X_H(a(t))\end{aligned}$$

■

Remark. The above way of finding solutions of the system (1.1) is very commonly used in the wave and Schrödinger (either linear on non-linear) equations.

First, let us consider the case when $G = S^1$. Denote ω as an element in **R** which is the Lie algebra of S^1 and $L_\omega = H - \omega J$. By (1.2), $\mathrm{d}L_\omega(x_\omega) = 0$ and so the Hessian of L_ω, $\mathrm{d}^2L_\omega(x_\omega)$ is well-defined.

Consider the quadratic form induced from $\mathrm{d}^2L_{\omega_0}(x_{\omega_0})$ on $T_{x_0}M$. From the symmetry, $\mathrm{d}^2L_{\omega_0}(\xi_{S^1}(x_{\omega_0})) = 0$. It is easy to check, at least in finite dimension that if $\mathrm{d}^2L_{\omega_0}(x_{\omega_0})$ is positive definite in a complementry space of the span of $\xi_{S^1}(x_{\omega_0})$, then the above solution is orbitally stable. However, if $\mathrm{d}^2L_{\omega_0}(x_{\omega_0})$ has some negative eigenspace, it is not clear at all whether the orbit is stable or not.

In Grillakis et al. [3], a simple criterion is noticed for stability and instability which has been implicitly used in many literatures on non-linear stability of the wave and Schrödinger equations when $\mathrm{d}^2L_{\omega_0}(x_{\omega_0})$ has only one simple negative (resp. positive) eigenvalue and all the others are positive (resp. negative) except the above trivial zero eigenvalue.

THEOREM 1. (See [3]). *Assume that there is a smooth family of relative equilibria x_ω depending on ω near ω_0. Assume*

$$\mathrm{d}J(x_{\omega_0}) \neq 0.$$

where J is the moment mapping. Define a function $\mathrm{d}(\omega) = H(x_\omega) - \omega J(x_\omega)$

near ω_0 *and assume that* $\mathrm{d}^2 L_{\omega_0}(x_{\omega_0})$ *has one dimensional negative and obvious one dimensional zero space. Then,*

i) if $d''(\omega_0) > 0, x_{\omega_0}$ *is orbitally stable.*
ii) if $d''(\omega_0) < 0, x_{\omega_0}$ *is orbitally unstable.*

Remark. i) One interesting feature of this theorem is that it does not involve the eigenvalue analysis of the linearized Hamiltonian vector field at x which is usually necessary to check the linear or non-linear stability, but it only involves the analysis of the Hessian and an auxilliary function of one variable.

ii) We can also apply this criterion to the case where the Hessian has only one positive eigenvalue by considering $-L_\omega$ and $-d''(\omega)$.

iii) The hypothesis that we have a smooth family of relative equilibria is not restrictive at all since it will be automatically satisfied if we have one relative equilibrium x_{ω_0} and $\mathrm{d}^2 L_{\omega_0}(x_{\omega_0})$ is invertible on S^1-quotient.

Next, let us consider the case for larger group stated in the beginning of this section. Assume that there is a smooth family of solutions $x(\xi)$ of the equation

$$\mathrm{d}H(x) - < \xi, \mathrm{d}J(x) > = 0$$

in a neighborhood of ξ_0 in g_μ where $x(\xi_0) = x_0$ is the given relative equilibrium.

Define the function $B(\xi)$ on g_μ near ξ_0 by

$$B(\xi) = H(x(\xi)) - < \xi, J(x(\xi)) >$$

and define L_ξ on M by

$$L_\xi(x) = H(x) - < \xi, J(x) >.$$

Here, since we only need this function in a neighborhood of x_0, the local existence of J is sufficient for our purpose. Then we have the following generalization of theorem 1.

THEOREM 2. *Assume that*

i) the (negative) inertia index of $\mathrm{d}^2 L_{\xi_0}(x_0)$ *is less than equal to dim* g_μ,
ii) the inclusion $g_\mu \cdot x_0 \subset \ker \mathrm{d}^2 L_{\xi_0}(x_0)$ *is an equality,*
iii) $\mu = J(x_0)$ *is in a regular (maximal) leaf,*
iv) $\mathrm{d}^2 B(\xi_0)$ *is positive deginite on* g_μ.

Then, the relative equilibrium x_0 *is (orbitally) stable.*

Remark. i) As will be seen in the course of proof, the hypotheses i) and iv) together imply the equality in the hypothesis i) and so theorem 2 implies i) of theorem 1.

ii) If the group G is abelian, every μ is regular and so the hypothesis iii) is automatically satisfied.

iii) We suspect that if $d^2B(\xi_0)$ is still nondegenerate but has odd number of negative eigenvalues, then the equilibrium is (Lyapunov) unstable, but we could not prove yet.

Now we summarize the content of the paper. We give the simplified proof of theorem 1 in the finite dimensional case essentially following the idea of [3] in section 2 for those who feel uncomfortable with the technicalities arising from the infinite dimensions as in [3], apply the criterion to the stability questions of the free rigid body and coupled planar rigid bodies in section 3 and then give the proof of theorem 2 in the last section.

2. PROOF OF THE THEOREM 1

We will prove the first part of theorem 1. First, let us derive several a priori identities from the definition and assumptions. By definition of x_ω, we have the following equality,

$$dL_\omega(x_\omega) = dH(x_\omega) - \omega\, dJ(x_\omega) = 0. \tag{2.1}$$

Differentiating this equation with respect to ω, we get the following,

$$0 = d^2H(x_\omega)\cdot\frac{dx_\omega}{d\omega} - \omega\, d^2J(x_\omega)\cdot\frac{dx_\omega}{d\omega} - dJ(x_\omega).$$

In other words,

$$\begin{aligned} d^2L_\omega(x_\omega)\cdot\frac{dx_\omega}{d\omega} &= d^2H(x_\omega)\cdot\frac{dx_\omega}{d\omega} - \omega\, d^2J(x_\omega)\cdot\frac{dx_\omega}{d\omega} \\ &= dJ(x_\omega). \end{aligned} \tag{2.2}$$

Strictly speaking, we have to introduce a connection to define the Hessian of H and J at x_ω and to have each term of middle of (2.2) make sense but the **LHS** (or **RHS**) itself does not depend on the connection and so the middle itself does not.

Now, let P be the positive eigenspace and η be the negative eigenvector of $d^2L_{\omega_0}(x_{\omega_0})$ of unit length with eigenvalue $-\lambda^2 < 0$ with respect to an S^1-invariant metric.

2.1. Stability

LEMMA 1. *Decompose orthogonally*

$$\operatorname{Ker} dJ(x_\omega) = \textit{span of}\,\{\xi_{S^1}(x_{\omega_0})\} \oplus Q.$$

where $\xi_{S^1}(x_{\omega_0})$ *is the vector field generated by* S^1. *Then* $\mathrm{d}^2L_{\omega_0}|_Q$ *is positive definite.*

Proof. Differentiating

$$\mathrm{d}(\omega) = H(x_\omega) - \omega J(x_\omega)$$

we get by (2.1)

$$(2.3)\quad \begin{aligned} d'(\omega) &= \mathrm{d}H(x_\omega)\cdot \frac{\mathrm{d}x_\omega}{\mathrm{d}\omega} - J(x_\omega) - \omega\,\mathrm{d}J(x_\omega)\cdot\frac{\mathrm{d}x_\omega}{\mathrm{d}\omega} \\ &= -J(x_\omega) \end{aligned}$$

since x_ω is a relative equilibrium. Differentiating again,

$$(2.4)\quad \begin{aligned} \mathrm{d}''(\omega) &= -\,\mathrm{d}J(x_\omega)\cdot\frac{\mathrm{d}x_\omega}{\mathrm{d}\omega} \\ &= -< \mathrm{d}^2L_\omega(x_\omega)\cdot\frac{\mathrm{d}x_\omega}{\mathrm{d}\omega}, \frac{\mathrm{d}x_\omega}{\mathrm{d}\omega} > \end{aligned}$$

from (2.2). From the assumption, $d''(\omega_0) > 0$, we have

$$(2.5)\quad < \mathrm{d}^2L_{\omega_0}\cdot\frac{\mathrm{d}x_\omega}{\mathrm{d}\omega}, \frac{\mathrm{d}x_\omega}{\mathrm{d}\omega} > \, < 0.$$

Decompose orthogonally

$$\frac{\mathrm{d}x_\omega}{\mathrm{d}\omega}(\omega_0) = a\eta + b\xi_{S^1} + p$$

for some $p \in P$ with $a \neq 0$ from (2.5). Substitute this back into (2.5) and get

$$(2.6)\quad -a^2\lambda^2 + < \mathrm{d}^2L_\omega p, p > \, < 0.$$

Now, let $v \in Q$, i.e. $v \perp \xi_{S^1}(x_\omega)$ and $v \in \operatorname{Ker} \mathrm{d}J(x_\omega)$ and write

$$v = a'\eta + p'$$

where $p' \in P$. By (2.2),

$$(2.7)\quad \begin{aligned} 0 &= < \mathrm{d}J(x_{\omega_0}), v > \\ &= < \mathrm{d}^2L_{\omega_0}(x_{\omega_0})\cdot\frac{\mathrm{d}x_\omega}{\mathrm{d}\bar{\omega}}, v > \\ &= < \mathrm{d}^2L_{\omega_0}(x_{\omega_0})(a\eta + b\xi_{S^1} + p), a'\eta + p' > \\ &= -aa'\lambda^2 + < \mathrm{d}^2L_\omega(x_\omega)p, p' >. \end{aligned}$$

Therefore,

$$\begin{aligned} < d^2 L_{\omega_0}(x_{\omega_0})v, v > &= - a'^2\lambda^2 + < d^2 L_{\omega_0} p', p' > \\ &\geqslant - a'^2\lambda^2 + \frac{< d^2 L_{\omega_0} p, p' >^2}{< d^2 L_{\omega_0} p, p >} \\ &> - a'^2\lambda^2 + \frac{(aa'\lambda^2)^2}{a^2\lambda^2} = 0. \end{aligned}$$

For the first inequality, we used the Schwarz' inequality since $d^2 L_\omega(x_{\omega_0})$ is symmetric and positive definite on P. For the second, we used (2.6) and (2.7). ■

One immediate consequence of this lemma is that the negative eigendirection is transversal to Ker $dJ(x_0)$.

Proof of the stability: Since J is conserved, Φ o J is conserved for any smooth function Φ on **R**. We are going to use the standard *Energy-Casimir method* [4]. First, we will choose Φ so that the function $H(x) + (\Phi \text{ o } J)(x)$ satisfies,

$$dH(x_{\omega_0}) + \Phi'(J(x_{\omega_0})) \cdot dJ(x_{\omega_0}) = 0,$$

i.e.

$$\Phi'(J(x_{\omega_0})) = - \omega_0.$$

Secondly, we require its second variation to be positive definite on $\{\xi_{S^1}(x_{\omega_0})\}^\perp$. However, the second variation is

$$\begin{aligned} &d^2 H(x_{\omega_0}) + \Phi'(J(x_{\omega_0}) \cdot d^2 J(x_{\omega_0} + \Phi''(J(x_{\omega_0})) \cdot dJ(x_{\omega_0}) \otimes dJ(x_{\omega_0}) \\ &= d^2 L_{\omega_0}(x_{\omega_0}) + \Phi''(J(x_{\omega_0})) \cdot dJ(x_{\omega_0}) \otimes dJ(x_{\omega_0}). \end{aligned}$$

We know by lemma 1 that $d^2 L(x_{\omega_0})|_Q$ is positive definite and adding the second term does not affect this fact since

$$dJ(x_{\omega_0}) = 0$$

in $Q \subset$ Ker $dJ(x_{\omega_0})$. Now, all we have to take care is the negative eigendirection. For that, choose $\Phi''(J(x_{\omega_0}))$ so that

$$\Phi''(J(x_{\omega_0})) > \frac{\lambda^2}{(dJ(x_{\omega_0})(\eta))^2} \ .$$

which is certainly possible. Therefore, x_{ω_0} is orbitally stable. ■

2.2. Instability

LEMMA 2. *If we have that* $d''(\omega_0) < 0$, *then there exists* u *such that*

$$< d^2 L_{\omega_0}(x_{\omega_0})u, u > \, < 0$$

and $dJ(x_{\omega_0})u = 0$.

Proof. Let η_{ω_0} be the unit eigenvector with negative eigenvalue, η_ω be the parallel translate along the 1-parameter curve x_ω and let $\gamma_\omega(s)$ be the geodesic with $\gamma_\omega(0) = x_\omega$ and $\gamma'_\omega(0) = \eta_\omega$. Consider the function $q(s, \omega) = J(\gamma_\omega(s))$. We have

$$\begin{aligned} \frac{\partial q}{\partial \omega}(0, \omega_0) &= dJ(x_{\omega_0}) \cdot \frac{\partial}{\partial \omega}\bigg|_{\omega = \omega_0} \gamma_\omega(0) \\ &= dJ(x_{\omega_0}) \cdot \frac{dx_\omega}{d\omega}\bigg|_{\omega = \omega_0} \\ &= - d''(\omega_0) > 0 \end{aligned} \tag{2.8}$$

by (2.4) and the definition of d. By the implicit function theorem, there is a smooth function $\omega(s)$ such that

$$J(\gamma_{\omega(s)}(s)) = J(x_{\omega_0}), \qquad \omega(0) = \omega_0 \tag{2.9}$$

If $u = \dfrac{d}{ds}\bigg|_{s=0} \gamma_{\omega(s)}(s)$, differentiating (2.9) yields

$$dJ(x_{\omega_0}) \cdot u = 0. \tag{2.10}$$

Moreover, $u \neq 0$ since

$$\begin{aligned} u &= \frac{d}{ds}\bigg|_{s=0} \gamma_{\omega(s)}(s) \\ &= \gamma'_{\omega_0}(0) + \frac{\partial}{\partial \omega}\bigg|_{\omega = \omega_0} \gamma_\omega(0) \cdot \omega'(0) \\ &= \eta_{\omega_0} + \omega'(0) \frac{dx_\omega}{d\omega}(\omega_0). \end{aligned}$$

Indeed, from the assumption that $d''(\omega_0) < 0$, it follows that

$$< d^2 L_{\omega_0}(x_{\omega_0}) \cdot \frac{dx_\omega}{d\omega}(\omega_0), \frac{dx_\omega}{d\omega}(\omega_0) > \, > 0 \text{ from (2.4).}$$

Now,

$$\langle d^2 L_{\omega_0}(x_{\omega_0})u, u\rangle = \frac{d^2}{ds^2}\Big|_{s=0} L_{\omega(s)}(\gamma_{\omega(s)}(s))$$

(2.11)
$$= \frac{d^2}{ds^2}\Big|_{s=0} H(\gamma_{\omega(s)}(s))$$

(since J is constant along $\gamma_{\omega(s)}\cdot$)

$$= \frac{d^2}{ds^2}\Big|_{s=0} \{H(\gamma_{\omega(s)}(s)) - H(x_{\omega_0})\}.$$

Note that $H(\gamma_{\omega(s)}(s)) - H(x_{\omega(0)})$ vanishes up to order 2 since $\gamma_{\omega(0)}(0) = x_{\omega_0}$ and

$$\frac{d}{ds}\Big|_{s=0} H(\gamma_{\omega(s)}(s)) = dH(x_{\omega_0})\cdot \frac{d}{ds}\Big|_{s=0} \gamma_{\omega(s)}(s)$$
$$= \omega_0 \, dJ(x_{\omega_0})\cdot u = 0$$

by (2.10). Therefore,

$$\langle d^2 L_{\omega_0}(x_{\omega_0})u, u\rangle = \frac{d^2}{ds^2}\Big|_{s=0} \{H(\gamma_{\omega(s)}(s)) - H(x_{\omega_0})\}$$

(2.12)
$$= \lim_{s\to 0} \frac{1}{2s^2}\,(H(\gamma_{\omega(s)}(s)) - H(x_{\omega_0})).$$

Now,

$$L_\omega(\gamma_\omega(s)) = L_\omega(x_\omega) + s\, dL_\omega(x_\omega)\eta_\omega + \frac{1}{2}\langle d^2 L_\omega(x_\omega)\eta_\omega, \eta_\omega\rangle + o(s^2)$$

$$= L_\omega(x_\omega) + \frac{1}{2} s^2 \langle d^2 L_\omega(x_\omega)\eta_\omega, \eta_\omega\rangle + o(s^2)$$

Therefore,

$$H(\gamma_{\omega(s)}) = \omega(s) J(\gamma_{\omega(s)}(s)) + d(\omega(s))$$
$$+ \frac{1}{2} s^2 \langle d^2 L_{\omega(s)}(x_{\omega(s)})\eta_{\omega(s)}, \eta_{\omega(s)}\rangle + o(s^2)$$

(2.13)
$$= \omega(s) J(x_{\omega_0}) + d(\omega(s))$$
$$+ \frac{1}{2} s^2 \langle d^2 L_{\omega(s)}(x_{\omega(s)})\eta_{\omega(s)}, \eta_{\omega(s)}\rangle + o(s^2)$$

from the definition (2.9) of $\omega(s)$. Since $d''(\omega_0) < 0$,

$$\begin{aligned} d(\omega(s)) &< d(\omega_0) + (\omega(s)) - \omega_0)\, d'(\omega_0) \qquad \text{for small } s, \\ &= H(x_{\omega_0}) - \omega(s) J(x_{\omega_0}) \end{aligned}$$

by the definition of d and (2.3). Substituting this into (2.12), we get

$$H(\gamma_{\omega(s)}(s)) < H(x_{\omega_0}) + \frac{1}{2} s^2 < d^2 L_{\omega(s)}(x_{\omega(s)}) \cdot \eta_{\omega(s)}, \eta_{\omega(s)} > + o(s^2). \tag{2.14}$$

Moreover, we may assume that

$$< d^2 L_{\omega(s)}(x_{\omega(s)}) \eta_{\omega(s)}, \eta_{\omega(s)} > \leqslant \frac{1}{2} < d^2 L_{\omega_0} \eta_{\omega_0}, \eta_{\omega_0} > < 0 \tag{2.15}$$

by continuity. Now, substitute (2.14) into (2.12) and use (2.15) to get

$$< d^2 L_{\omega_0}(x_{\omega_0}) u, u > \leqslant \frac{1}{2} < d^2 L_{\omega_0} \eta_{\omega_0}, \eta_{\omega_0} > < 0.$$

■

One immediate consequence of this Lemma is that on the reduced space, the reduced Hamiltonian has one negative eigenvalue and all the other positive eigenvalues.

Now, the instability theorem follows from the following general fact which actually proves the spectral instability on the reduced space.

LEMMA 3. *Let M be a symplectic manifold and H be a Hamiltonian function. Assume that* $dH(x) = 0$, $d^2 H(x)$ *has odd number of negative eigenvalues, and all the other eigenvalues are positive.*

Then x is spectrally unstable and so non-linearly (or Lyapunov) unstable.

Proof. Choose Darboux coordinates near x. Then $DX_H(x)$ can be written as $\Omega^{-1} \cdot d^2 H(x)$ where

$$\Omega^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and $d^2 H(x)$ is considered as a map from $T_x M$ to $T_x^* M$. Therefore,

$$\det DX_H(x) = \det(\Omega^{-1} \cdot d^2 H(x)) = \det \Omega^{-1} \cdot \det(d^2 H(x)) < 0$$

since $\det \Omega^{-1} > 0$ and $\det d^2 H(x) < 0$ from the assumption. Therefore, $DX_H(x)$ should have at least one (and so two) real eigenvalues and so is spectrally unstable. For, if $DX_H(x)$ has no real eigenvalue, then $\det DX_H(x)$ must be positive since the eigenvalues of $DX_H(x)$ are grouped as $\{\lambda, \bar{\lambda}\}$ when λ is purely imaginary or $\{\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}\}$ when λ is not purely imaginary. In any case, the products of these pairs or quadruples are all positive and so det $DX_H(x)$ would be positive. ∎

The proof of the instability theorem is now an easy consequence of lemma 2 and 3 by working on the reduced space.

Remark. It may be interesting to study if the instability result in [3] is due to spectral instability like here in the finite dimensional context.

2.3. Refinement in the Poisson Category

Before we apply theorem 1 to planar coupled rigid bodies, we adapt our theorem to Poisson manifolds with codimension one symplectic leaves at regular points. In fact, our theorem will be just a special case of this adaptation since M/S^1 has a natural Poisson structure near regular points.

Let P be a Poisson manifold with codimension one symplectic leaves at regular points, H be a Hamiltonian function and p be an equilibrium. Assume that p is a regular point.

Let C be a given Casimir function whose level sets are symplectic leaves near p. Then, by the Lagrange multiplier theorem

$$\mathrm{d}H(p) - \omega_0 \, \mathrm{d}C(p) = 0$$

for some $\omega_0 \in \mathbf{R}$. Assume that we have a smooth family of equilibria p_ω near $p_{\omega_0} = p$ and define L_{ω_0} by

$$L_{\omega_0}(p) = H(p) - \omega_0 C(p).$$

Assume that $\mathrm{d}^2 L_{\omega_0}(p)$ has one negative eigenvalue and all the others are positive. If we define

$$\mathrm{d}(\omega) = H(p_\omega) - \omega C(p_\omega),$$

then we have the same criterion for the stability in this Poisson category as in our main theorem.

3. APPLICATIONS

3.1. The free rigid body.

Here, we will follow the material in [4, p. 11 - 13].

The equations of free rigid body motion are given by

(3.1) $$m' = \frac{dm}{dt} = m \times \omega$$

where ω is the angular velocity and m is the angular momentum, both viewed in the body coordinate system. The relation between m and ω is given by

$$m_i = I_i \omega_i \qquad i = 1, 2, 3$$

where $I = (I_1, I_2, I_3)$ is the diagonalized moment of inertia tensor. We can consider this system as «Hamiltonian» in the Lie-Poisson structure in $R^3(\cong so(3))$ with the Hamiltonian $H(m) = \frac{1}{2} \Sigma_{i=1}^{3} \frac{m_i^2}{I_i}$. Note that the regular leaves of $so(3)$ are concentric spheres of codimension one and so we may apply our criterion to check the stability. The Casimirs are the functions of the form $C_\phi(m) = \phi\left(\frac{|m|^2}{2}\right)$.

Let us choose $C = (\frac{1}{2} \mid m \mid^2)^2$ where we choose this rather than $\frac{1}{2} \mid m \mid^2$ so that the Hessian of the corresponding L_ω be non-degenerate. Now consider the equilibrium $e = (1, 0, 0)$. Then,

(3.2) $$dH(e) = \left(\frac{1}{I_1}, 0, 0\right)$$

(3.3) $$dC(e) = (2, 0, 0)$$

Set $L_\omega = H - \omega C$ and $\omega_0 = \frac{1}{2I_1}$. Choose $e_\omega = \left(\frac{1}{\sqrt{2\omega I_1}}, 0, 0\right)$. Therefore,

$$\begin{aligned} d(\omega) &= H(e_\omega) - \omega C(e_\omega) \\ &= \frac{1}{4\omega I_1^2} - \frac{1}{8\omega I_1^2} \\ &= \frac{1}{8\omega I_1^2} \end{aligned}$$

Differentiating this twice we get, $d''(\omega) = \frac{1}{4I_1^2\omega^3}$ and so,

(3.4) $$d''(\omega_0) = 2I_1 > 0$$

from the above. Now, consider the Hessian $d^2L_{\omega_0}(e)$. By simple computations, we get

$$< d^2L_{\omega_0}(e) \cdot \delta m, \delta m > = \left(\frac{1}{I_2} - \frac{1}{I_1}\right)(\delta m_2)^2 + \left(\frac{1}{I_3} - \frac{1}{I_1}\right)(\delta m_3)^2 - \frac{1}{I_1}(\delta m_1)^2$$

since $L_{\omega_0} = H - \dfrac{1}{2I_1} C$.

i) The case $I_1 > I_2, I_1 > I_3$:

In this case, $d^2 L_{\omega_0}(e)$ has two positive eigenvalues and one negative eigenvalue. Since $d''(\omega_0) = 2I_1 > 0$ from (3.4), we can conclude that (1, 0, 0) is stable by our theorem.

ii) The case $I_2 < I_1 < I_3$ (or $I_3 < I_1 < I_2$):

In this case, $d^2 L_{\omega_0}$ has two negative eigenvalues and one positive eigenvalue. Hence, by (3.4) and the remark ii) after our theorem 1, this equilibrium is unstable.

iii) The case $I_1 < I_2, I_1 < I_3$:

In this case, $d^2 L_{\omega_0}$ is negative definite and so stable without referring to our criterion.

3.2. Stability of Coupled Planar Rigid Bodies

In this section, we are going to follow the notations of [8, 9]. We restrict ourselves to the case $\kappa > 0$, $\tau > 0$, i.e. the case that the center of mass of the second body is aligned with the two hinge points and between the two points in the reference configuration. (See Figure 1).

After we reduce the system by translation of center of mass and the total angular momentum, we get a system on $T^*(S^1 \times S^1 \times S^1)/S^1$ which is a Poisson manifold whose symplectic leaves are of codimension one. The total angular momentum Φ and its functions are the Casimir functions on this manifold. (This can be easily seen from the definition of quotient bracket and the fact that any function on this quotient can be considered as an S^1-invariant function and brackets of Φ with such functions in $T^*(S^1 \times S^1 \times S^1)$ vanish by Nöther's theorem).

As in [8], we parametrize $T^*(S^1 \times S^1 \times S^1)/S^1$ by $(\theta_{21}, \theta_{32}, \mu_1, \mu_2, \mu_3))$ and the we can write the Hamiltonian $H(\theta, \mu)$ as follows;

$$H(\theta, \mu) = \frac{1}{2} < \mu, J^{-1}\mu > \left(= \frac{1}{2} < \omega, J\omega > \right)$$

where J is the associated metric on $S^1 \times S^1 \times S^1$ which depends on system parameters. And the total angular momentum is

$$\Phi(\theta, \mu) = \mu_1 + \mu_2 + \mu_3.$$

The relative equilibria will be described by the equation

$$dH - \omega \, d\Phi = 0$$

for some ω. The above equation can be decomposed into following two equations;

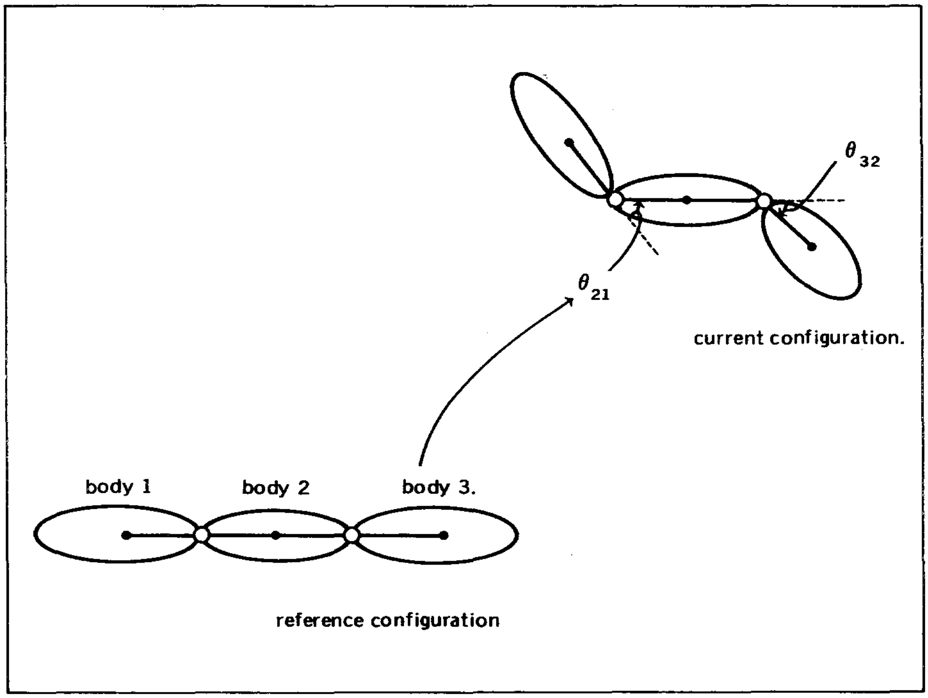

Figure 1.

(3.5) $$d_\mu H - \omega d_\mu \Phi = 0$$

(3.6) $$d_\theta H = 0$$

From (3.6),

$$0 = d_\theta H = \frac{1}{2} < \mu, d_\theta J^{-1} \mu >$$

(3.7)

$$- \frac{1}{2} < \mu, J^{-1} \cdot d_\theta J \cdot J^{-1} \mu >$$

From (3.5),

$$J^{-1} \mu = \omega(1, 1, 1), \text{ i.e. } \mu = \omega J(1, 1, 1)$$

Substituting this into (3.7), we get the equation

$$\omega^2 < (1, 1, 1), \ d_\theta J(1, 1, 1) > = 0$$

This equation can be written as follows (See [8]);

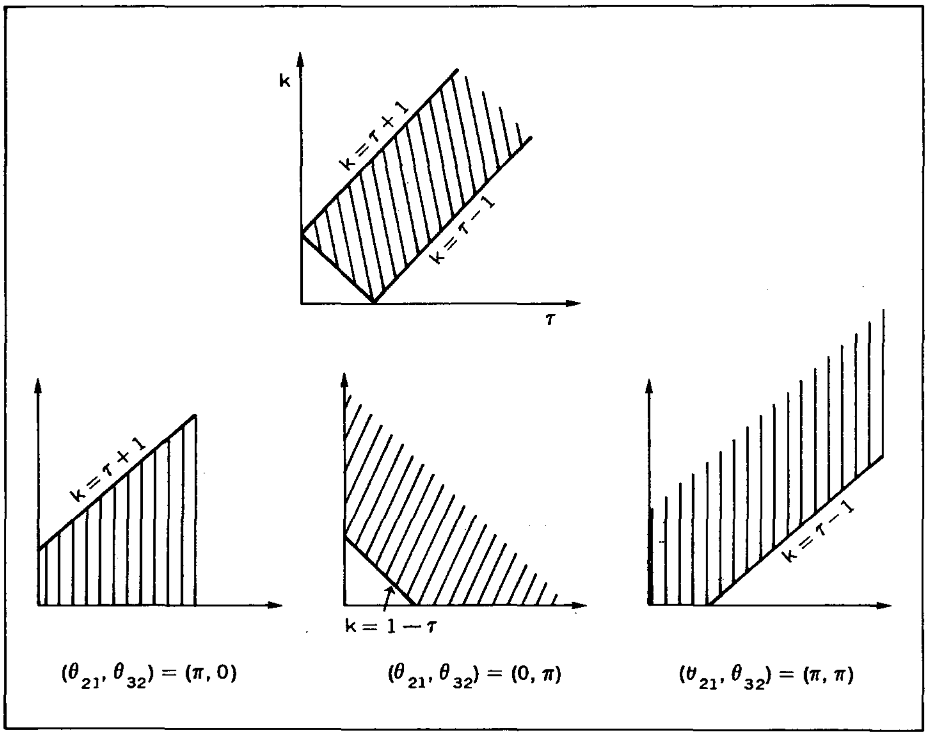

Figure 2.

$$\text{(3.8)} \qquad \begin{cases} \sin(\theta_{21} + \theta_{32}) = -\tau \sin(\theta_{21}) \\ \sin(\theta_{32}) = \kappa \sin(\theta_{21}) \end{cases}$$

In [8, 9], it was shown that if $\kappa \leqslant |\tau - 1|$ or $\kappa \geqslant |\tau + 1|$, then (3.8) has only the solutions

$$(\theta_{21}, \theta_{32}) = (0, 0), (\pi, 0), (0, \pi) \text{ and } (\pi, \pi)$$

and if $|\tau - 1| < \kappa < |\tau + 1|$, (3.8) has two more solutions besides the above as follows:

i) $(\theta_{21}, \theta_{32}) = (0, 0)$; the Hessian of H is positive definite and so this equilibrium is stable.

ii) $(\theta_{21}, \theta_{32}) = (\pi, 0)$; the Hessian has just one negative eigenvalue if $\kappa \leqslant \tau + 1$.

iii) $(\theta_{21}, \theta_{32}) = (0, \pi)$; the Hessian has just one negative eigenvalue if $\kappa \geqslant 1 - \tau$.

iv) $(\theta_{21}, \theta_{32}) = (\pi, \pi)$; the Hessian has just one negative eigenvalue if $\kappa \geqslant \tau - 1$.

Therefore, we can apply our instability theorem to each of these cases. All we have to do is to check the sign of $d''(\omega)$ where

$$d(\omega) = H(x_\omega) - \omega\Phi(x_\omega) \text{ for } x_\omega = (\theta_\omega, \mu_\omega).$$

Here, we know (see [7]) that

$$\theta_\omega = \text{const.} \qquad \text{which is among the above}$$

$$\mu_\omega = J(\theta_\omega) \cdot < \omega, \omega, \omega > = \omega J(\theta_\omega) < 1, 1, 1 >$$

Now,

$$H(x_\omega) = \frac{1}{2} < \mu_\omega, J^{-1}(\theta_\omega)\mu_\omega >$$

$$= \frac{1}{2} \omega^2 < (1, 1, 1), J(\theta_\omega)(1, 1, 1), >$$

$$\Phi(x_\omega) = \mu_{\omega,1} + \mu_{\omega,2} + \mu_{\omega,3} = < \mu_\omega, (1, 1, 1) >$$

$$= \omega < (1, 1, 1), J(\theta_\omega)(1, 1, 1) >$$

Therefore,

$$\mathrm{d}(\omega) = -\frac{1}{2} \omega^2 < (1, 1, 1), J(\theta_\omega)(1, 1, 1) >$$

and so,

$$d''(\omega) = -\frac{1}{2} < (1, 1, 1), J(\theta_\omega)(1, 1, 1) > < 0$$

since θ_ω is constant as ω varies and J is positive definite. Hence, the relative equilibria $\theta_\omega = (\pi, 0)$, $(0, \pi)$ and (π, π) are unstable in each region as above respectively. In the other region, these relative equilibria have two negative eingenvalues and so we cannot apply our theorem to these cases.

Remark. Although we cannot apply our theorem to the other cases where the Hessian has two negative eigenvalues, we were able to prove these equilibria are always linearly (in fact, spectrally) unstable in a different method (See [9]).

4. GENERLIZATION FOR LARGER GRCUPS

4.1. Proof of the theorem 2

As in the section 2, we get the following identities:

(4.1) $$B(\xi) = H(x(\xi)) - < \xi, J(x(\xi)) >$$

from the definition of B,

(4.2) $$\mathrm{d}L_\xi(x(\xi)) = \mathrm{d}H(x(\xi)) - < \xi, \mathrm{d}J(x(\xi)) > = 0$$

by the definition of $x(\xi)$ and L_ξ,

(4.3) $$\begin{aligned} \mathrm{d}^2 L_\xi(x(\xi)) \cdot \delta x(\xi) &= \mathrm{d}^2 H(x(\xi)) \cdot \delta x(\xi) - < \xi, \mathrm{d}^2 J(x(\xi)) > \cdot\, \delta x(\xi) \\ &= \mathrm{d}J(x(\xi)) \end{aligned}$$

by differentiating (4.1) with respect to ξ, where $\delta x(\xi) : g_\mu \to T_{x(\xi)}M$ is defined by

(4.4) $$\delta x(\xi) \cdot \eta = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} x(\xi + t\eta).$$

Now, set

$$P = \text{positive eigenspace of } \mathrm{d}^2 L_{\xi_0}(x_0)$$

$$K = \text{kernel of } \mathrm{d}^2 L_{\xi_0}(x_0) = g_\mu \cdot x_0$$

$$N = \text{negative eigenspace of } \mathrm{d}^2 L_{\xi_0}(x_0)$$

and denote Π_P, Π_K and Π_N at the orthogonal projections onto these subspaces of $T_{x_0}M$ respectively.

LEMMA 4. *Decompose Ker* $\mathrm{d}J(x_{\xi_0})$ *as an orthogonal direct sum*

$$\text{Ker } \mathrm{d}J(x_{\xi_0}) = K \oplus Q.$$

Then $\mathrm{d}^2 L_{\xi_0}|_Q$ *is positive definite under the hypotheses of theorem 2.*

Proof. By differentiating (4.1) and using (4.2)

(4.5) $$\begin{aligned} \mathrm{d}B(\xi) &= \mathrm{d}H(x(\xi)) \cdot \delta x(\xi) - J(x(\xi)) - < \xi, \mathrm{d}J(x(\xi)) \cdot \delta x(\xi) > \\ &= - J(x(\xi)) \end{aligned}$$

Defferentiating again,

(4.6) $$\begin{aligned} \mathrm{d}^2 B(\xi) &= - \mathrm{d}J(x(\xi)) \cdot \delta x(\xi) \\ &= - < \mathrm{d}^2 L_\xi(x(\xi)) \cdot \delta x(\xi), \delta x(\xi) > \end{aligned}$$

from (4.3) for the second equality. From the hypothesis in theorem 2, $\mathrm{d}^2 B(\xi_0)$ is positive definite on g_μ and so the quadratic form $< \mathrm{d}^2 L_{\xi_0}(x_{\xi_0}) \cdot \delta x(\xi_0), \delta x(\xi_0) >$ is negative definite on g_μ, i.e.

$$\begin{aligned} &< \mathrm{d}^2 L_{\xi_0}(x_{\xi_0}) \cdot \Pi_N \delta x(\xi_0)\eta, \Pi_N \delta x(\xi_0)\eta > \\ &\qquad + < \mathrm{d}^2 L_{\xi_0}(x_0) \cdot \Pi_P \delta x(\xi_0)\eta, \Pi_P \delta x(\xi_0) > < 0 \end{aligned}$$

for all $0 \neq \eta \in g_\mu$. In particular, the subspace $\delta x(\xi_0) \cdot g_\mu \subset T_{x(\xi_0)}M$ is transversal to $P \oplus K$ since $\dim N \leqslant \dim g_\mu$. (Here, in fact, we have $\dim N = \dim g_\mu$). Now, let $v \in Q$, i.e. $v \perp K$ and $v \in \operatorname{Ker} dJ(x(\xi_0))$ and write $v = \Pi_N v + \Pi_P v$. Then by (4.3),

$$
\begin{aligned}
0 &= < dJ(x_{\xi_0}), v > \\
&= < d^2 L_{\xi_0}(x_0) \cdot \delta x(\xi_0), v > \qquad (4.8)\\
&= < d^2 L_{\xi_0}(x(\xi_0)) \cdot \Pi_P \delta x(\xi_0), \Pi_P v > + < d^2 L_{\xi_0}(x_0) \cdot \Pi_N \delta x(\xi_0), \Pi_N v >
\end{aligned}
$$

Since $\delta x(\xi_0) \cdot g_\mu$ is transversal to $P \oplus K$, we can find an $\eta \in g_\mu$ such that

$$\Pi_N \delta x(\xi_0) \cdot \eta = \Pi_N v \qquad (4.9)$$

for a given v. Then,

$$
\begin{aligned}
< d^2 L_{\xi_0}(x_0) v, v > &= < d^2 L_{\xi_0}(x_0) \cdot \Pi_N v, \Pi_N v > + < d^2 L_{\xi_0}(x_0) \cdot \Pi_P v, \Pi_P v > \\
&\geqslant < d^2 L_{\xi_0}(x_0) \cdot \Pi_N v, \Pi_N v > + \frac{< d^2 L_{\xi_0}(x_0) \cdot \Pi_P v, \Pi_P \delta x(\xi_0)\eta >^2}{< d^2 L_{\xi_0}(x_0) \cdot \Pi_P \delta x(\xi_0)\eta, \Pi_P \delta x(\xi_0)\eta >} \\
&> < d^2 L_{\xi_0}(x_0) \cdot \Pi_N v, \Pi_N v > - \frac{< d^2 L_{\xi_0}(x_0) \cdot \Pi_P v, \Pi_P \delta x(\xi_0)\eta >^2}{< d^2 L_{\xi_0}(x_0) \cdot \Pi_N \delta x(\xi_0)\eta, \Pi_N \delta x(\xi_0)\eta >} \\
&= < d^2 L_{\xi_0}(x_0) \cdot \Pi_N v, \Pi_N v > - \frac{< d^2 L_{\xi_0}(x_0) \cdot \Pi_N v, \Pi_N \delta x(\xi_0)\eta >^2}{< d^2 L_{\xi_0}(x_0) \cdot \Pi_N \delta x(\xi_0)\eta, \Pi_N \delta x(\xi_0)\eta >} \\
&= 0.
\end{aligned}
$$

Here, we used Schwarz' inequality for the first inequality, (4.7) for the second inequality, (4.8) for the second equality and (4.9) for the last equality. ∎

Proof of the theorem 2. Since J is (coadjoint) equivariant, $\Phi \circ J$ is conserved for any coadjoint invariant, i.e. Casimir function Φ on g^*. We can choose Φ so that

$$d\Phi(J(x_0)) = d\Phi(\mu) = -\xi_0$$

which is possible since the leaf containing μ is regular. (This is one place where we used this hypothesis). The second variation of $H + \Phi \circ J$ is given as

$$d^2 L_{\xi_0}(x_0) + < d^2 \Phi(J(x_0) \cdot dJ(x_0), dJ(x_0) >.$$

As in section 2, we have only to choose Φ so that

$$< d^2 \Phi(J(x_0) \cdot dJ(x_0), dJ(x_0) > |_N > - d^2 L_{\xi_0}(x_0)|_N. \qquad (4.10)$$

By (4.3), we know that

$$\mathrm{d}J(x_0) = \mathrm{d}^2 L_{\xi_0}(x_0) \cdot \delta x(\xi_0).$$

Now, consider $\mathrm{d}J(x_0)$ as an element in $T^*_{x_0}M \otimes g^*$ i.e. either as a map from g to $T^*_{x_0}M$ or $T_{x_0}M$ to g^*. Then,

$$\begin{aligned} &\mathrm{d}J(x_0)|_{g_\mu} \pitchfork P \oplus K \\ &\mathrm{d}J(x_0)|_N \pitchfork g_\mu^\perp \end{aligned} \tag{4.11}$$

by the remark after (4.7). Moreover, the possible maximal rank of $\mathrm{d}^2\Phi(\mu)$ is $\dim g_\mu$ since μ is in a regular leaf (this is another place where we used this hypothesis) and we can choose Φ so that

$$\operatorname{Ker} \mathrm{d}^2\Phi(\mu) = g_\mu^\perp (\cong T_\mu O_\mu) \tag{4.12}$$

By (4.11) and (4.12), it is possible for $< \mathrm{d}^2\Phi(\mu) \cdot \mathrm{d}J(x_0) > |_N$ to dominate $-\mathrm{d}^2 L_{\xi_0}(x_0)|_N$ for a suitable choice of Φ. ■

4.2. Poisson version

Let P be a Poisson manifold and $p_0 \in P$ an equilibrium in a regular leaf. Let $C_1, \ldots, C_k$ be local Casimirs near p_0 which are maximally independent. Write $C = (C_1 \ldots C_k)$. Let

$$\mathrm{d}H(p_0) - < \xi_0, C(p_0) > = 0.$$

where $\xi_0 \in \mathbf{R}^k$, and assume that we have a smooth family $p(\xi)$ near $\xi = \xi_0$ and with $p(\xi_0) = p_0$ which satisfies the equation

$$\mathrm{d}H(p) - < \xi, \mathrm{d}C(p) > = 0$$

Define $L_\xi(p) = H(p) - < \xi, C(p) >$ and $B(\xi) = H(p(\xi)) - < \xi, C(p(\xi)) >$. Then, we have the following Poisson version of theorem 2.

THEOREM 2′. *Assume that*

i) $\mathrm{d}^2 L_{\xi_0}(p_0)$ *has negative inertia index* $\leqslant k$

ii) $\mathrm{d}^2 B(\xi_0)$ *is positive definite.*

Then the equilibrium p_{ξ_0} *is stable.*

Remark. i) It may be interesting to find a similar criterion for the equilibria in the singular leaf. It seems that this is related with transverse Poisson structures.

ii) It can be easily seen that we can replace Casimir functions by any conserved quantities in theorem 2′. This allows us to have the similar criterion for an equilibrium in a singular leaf if it has some extra symmetries.

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Manuscript received: August 17, 1987.